

Existence and non-existence of the non-central Wishart distributions.

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Abstract

The problem considered in this paper is to find when the non-central Wishart distribution, defined on the cone $\overline{\mathcal{P}_d}$ of semi positive definite matrices of order d and with a real valued shape parameter, exists. We reduce this problem to the problem of existence of the measures $m(n, k, d)$ defined on $\overline{\mathcal{P}_d}$ and with Laplace transform $(\det s)^{-n/2} \exp \operatorname{tr}(s^{-1}w)$ where n is an integer and where $w = \operatorname{diag}(0, \dots, 0, 1, \dots, 1)$ has order d and rank k . We compute $m(d-1, d, d)$ and we show that neither $m(d-2, d, d)$ nor $m(d-2, d-1, d)$ exist. This proves a conjecture of E. Mayerhofer.

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ABBREVIATED TITLE: Non-central Wishart distributions.

1 Introduction

The non-central Wishart distribution is traditionally defined as the distribution of the random symmetric real matrix $X = Y_1 Y_1^* + \dots + Y_n Y_n^*$ where $Y_i \in \mathbb{R}^d$, $i = 1, \dots, n$ are independent Gaussian column vectors with the same non-singular covariance matrix Σ and respective means m_i , $i = 1, \dots, n$ not necessarily equal (here $*$ means transposition). For s in the open cone \mathcal{P}_d of positive definite symmetric matrices of order d and $w = m_1 m_1^* + \dots + m_n m_n^*$ in the closed cone of semi positive definite matrices $\overline{\mathcal{P}_d}$ one can readily derive the Laplace transform

$$\mathbb{E}(e^{-\operatorname{tr}(sX)}) = \frac{1}{\det(I_d + 2\Sigma s)^{n/2}} e^{-\operatorname{tr}(2s(I_d + 2\Sigma s)^{-1}w)}. \quad (1)$$

It is important to note that in this formula the rank of w is $\leq n$.

Exactly like the statistician who extends the familiar chi square distribution with n degrees of freedom to the gamma distribution with a continuous shape parameter, one is tempted to extend the values that the power of $\det(I_d + 2\Sigma s)$ can take in the above formula. The question is then: given $\Sigma \in \mathcal{P}_d$ and $w \in \overline{\mathcal{P}_d}$, for which values of $p > 0$ does there exist a probability distribution on $\overline{\mathcal{P}_d}$ for X such that for all $s \in \mathcal{P}_d$ we have

$$\mathbb{E}(e^{-\operatorname{tr}(sX)}) = \frac{1}{\det(I_d + 2\Sigma s)^p} e^{-\operatorname{tr}(2s(I_d + 2\Sigma s)^{-1}w)}? \quad (2)$$

Call this hypothetic distribution for X satisfying (2) a non-central Wishart distribution with parameters $(2p, w, \Sigma)$, or $NCW(2p, w, \Sigma)$ for short. This question was addressed in Letac and Massam

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(2008) where we claimed (in Proposition 2.3) that such a distribution exists if and only if p is in Λ_d with

$$\Lambda_d = \left\{ \frac{1}{2}, \frac{2}{2}, \dots, \frac{d-2}{2} \right\} \cup \left[\frac{d-1}{2}, \infty \right) \quad (3)$$

without any restriction on $\Sigma \in \mathcal{P}_d$ or $w \in \overline{\mathcal{P}}_d$. This statement was actually not quite proven: we considered as obvious that if p is in the part $\{\frac{1}{2}, \frac{2}{2}, \dots, \frac{d-1}{2}\}$ of Λ_d then $NCW(2p, w, \Sigma)$ does exist. The gap in our proof was kindly pointed out to us, in a private communication, by E. Mayerhofer who later showed in Mayerhofer (2010) that the statement was not only unproven, but false. More specifically Mayerhofer shows that, if $NCW(2p, w, \Sigma)$ exists, if $d \geq 3$ and if $2p$ is in $\{1, 2, \dots, d-2\}$, then $\text{rank } w \leq 2p+1$. He even conjectures that $\text{rank } w \leq 2p$ must hold. The aim of the present paper is to correct our mistake by giving a satisfactory necessary and sufficient condition of existence of $NCW(2p, w, \Sigma)$: in Section 2, we show that if $2p$ is in $\{1, 2, \dots, d-2\}$ then $\text{rank } w \leq 2p$, thus proving that the Mayerhofer conjecture is true. These questions are delicate and Mayerhofer (2011) uses a stochastic process valued in the set of symmetric matrices in order to prove his main statement (reformulated in Proposition 2.4 below). The methods of the present paper are simpler and necessitate only a careful study of the support of the measures in $\overline{\mathcal{P}}_d$ when these measures are obtained by convolution. The basic tool (Lemma 5.2) is the following. Let M_b be the set of positive measures concentrated on the matrices of rank b of $\overline{\mathcal{P}}_d$. It is not generally true that if $a+b \leq d$, $\mu \in M_a$ and $\nu \in M_b$, then $\mu * \nu \in M_{a+b}$, but it is true if either μ or ν is invariant by $x \mapsto uxu^{-1}$ for any orthogonal matrix u of order d . This result is the subject of Section 5. It is, however, not sufficient to prove Propositions 2.4 and 2.5: we need further information about the measure $m(d-1, d, d)$ on $\overline{\mathcal{P}}_d$ defined by its Laplace transform $(\det s)^{-(d-1)/2} \exp(\text{tr}(s^{-1}))$. We need to show that it has an absolutely continuous part. For this reason, Propositions 3.1 and 4.4 give a description of $m(1, 2, 2)$ and $m(d-1, d, d)$. We give there more details than strictly necessary but to describe the singular and absolutely continuous parts of $m(d-1, d, d)$, in Section 4, we make an interesting use of zonal polynomials. We are guided by an elementary study in Section 3 of the case $d=2$ which uses the Faà di Bruno formula only. We should also mention that the proper mathematical framework for this paper is that of Euclidean Jordan algebras rather than the linear spaces of real symmetric matrices and we make frequent references to Faraut and Korányi (1994) (henceforth abbreviated FK). But working in that framework might have obscured our statements without adding any insight: the extension of our results to Euclidean Jordan algebras is straightforward.

We would like to thank here E. Mayerhofer for pointing out our mistake and conjecturing the correct result, and J. Faraut for helping us with Lemma 4.1 below.

2 Reduction of the problem: the measures $m(2p, k, d)$

Let k be an integer such that $0 \leq k \leq d$. We consider the diagonal matrix $I(k, d)$ with its first $d-k$ diagonal terms equal to 0 and the last k equal to 1. For $p \in \Lambda_d$ we define the positive measure $m(2p, k, d)$ on $\overline{\mathcal{P}}_d$ such that for all $s \in \mathcal{P}_d$ we have

$$\int_{\overline{\mathcal{P}}_d} e^{-\text{tr}(sx)} m(2p, k, d)(dx) = \frac{1}{(\det s)^p} e^{\text{tr}(s^{-1}I(k, d))}. \quad (4)$$

Note that $m(2p, k, d)$ may or may not exist. Formula (22) below gives $m(1, 1, 1)$. For $2p > d-1$ formula (39) gives $m(2p, d, d)$. If k and n are integers such that $0 \leq k \leq n \leq d$, formula (7) gives $m(n, k, d)$. Finally $m(d-1, d, d)$ is computed in Sections 3 and 4. The paper will show that these examples are the only cases of existence. The following proposition links this measure $m(2p, k, d)$ with our initial existence problem.

Proposition 2.1. Let $\Sigma \in \mathcal{P}_d$, $w \in \overline{\mathcal{P}}_d$ and $p \in \Lambda_d$. Suppose that $\text{rank } w = k$. Then $NCW(2p, w, \Sigma)$ exists if and only if $m(2p, k, d)$ exists.

Proof. Assume that $m(2p, k, d)$ exists and let us show that $NCW(2p, w, \Sigma)$ exists. The proof is based on the following principle. Let μ be a positive measure on a finite dimensional real linear space E such that its Laplace transform $L_\mu(s) = \int_E e^{-\langle s, x \rangle} \mu(dx)$ is finite on some convex subset $D(\mu)$ of the dual space E^* with a non empty interior. Let a be a linear automorphism of E^* and let $b \in E^*$ such that $L_\mu(a(b)) < \infty$. Then there exists a probability $P(a, b)$ on E with Laplace transform $L_{P(a, b)}(s) = L_\mu(a(s + b))/L_\mu(a(b))$. This probability $P(a, b)$ is obtained in two steps: first take the image $\nu(dy)$ of $\mu(dx)$ by the map $x \mapsto a^*(x) = y$ where a^* is the adjoint of a . Its Laplace transform is $L_\nu(s) = L_\mu(a(s))$. The second step constructs $P(a, b)(dy)$ as the probability $e^{-\langle b, y \rangle} \nu(dy)/L_\mu(a(b))$.

Let us apply this principle to the case where $E = E^*$ is the Euclidean space of real symmetric matrices of order d with scalar product $\langle x, y \rangle = \text{tr}(xy)$ and where μ is $m(2p, k, d)$. Here $D(\mu) = \mathcal{P}_d$. We take $b = (2\Sigma)^{-1}$ and a to be the linear transformation $s \mapsto a(s) = qsq^*$ where q is an invertible matrix of order d such that

$$2(2\Sigma)^{-1}w(2\Sigma)^{-1} = q^{-1}I(k, d)(q^*)^{-1}. \quad (5)$$

We have $a^*(x) = q^*xq$. The distribution $P(a, b)$ is the noncentral Wishart $NCW(2p, w, \Sigma)$ since

$$\frac{L_\mu(a(s + b))}{L_\mu(a(b))} = \frac{1}{\det(I_d + 2\Sigma s)^p} e^{-\text{tr}(2s(I_d + 2\Sigma s)^{-1}w)}. \quad (6)$$

The verification of (6) is easily done by a calculation of trace using $\text{tr}(ab) = \text{tr}(ba)$ and (5):

$$\begin{aligned} & \text{tr} [((q^*)^{-1}(s + (2\Sigma)^{-1})^{-1}q^{-1} - (q^*)^{-1}(2\Sigma)q^{-1})I(k, d)] \\ &= \text{tr} [(s + (2\Sigma)^{-1})^{-1} - 2\Sigma]q^{-1}I(k, d)(q^*)^{-1}] = -\text{tr}(2s(I_d + 2\Sigma s)^{-1}w) \end{aligned}$$

The only thing left to prove is the existence of q satisfying (5). To see this, since the matrix $2(2\Sigma)^{-1}w(2\Sigma)^{-1}$ of $\overline{\mathcal{P}}_d$ has rank k , we write $2(2\Sigma)^{-1}w(2\Sigma)^{-1} = u\Delta u^*$ where

$$\Delta = \text{diag}(0, \dots, 0, \lambda_1^2, \dots, \lambda_k^2)$$

with $\lambda_i > 0$ and where u is an orthogonal matrix of order d . Taking $q = \text{diag}(1, \dots, 1, \lambda_1^{-1}, \dots, \lambda_k^{-1})u^*$ provides a solution of (5).

The proof of the converse follows similar lines. \square

Example: When $0 \leq k \leq n \leq d$ we can use the above principle for constructing $NCW(n, 2I(k, d), I_d)$ from $m(n, k, d)$. We take $q = I_d$ and $b = I_d/2$. Since a is the identity we have therefore

$$m(n, k, d)(dx) = 2^{dn/2} e^{2k} e^{\text{tr } x/2} NCW(n, 2I(k, d), I_d)(dx) \quad (7)$$

The next three propositions reformulate known facts in the language of the measures $m(2p, k, d)$.

Proposition 2.2. Let n and k be integers such that $0 \leq n, k \leq d$. The measure $m(n, k, d)$ exists for $0 \leq k \leq n \leq d$. Furthermore, the measure $m(d-1, d, d)$ exists.

Proof. Formula (7) provides an explicit form of $m(n, k, d)$. For $2p > d-1$ the probability $NCW(2p, I_d, I_d)$ exists as proved in Letac and Massam (2008) Proposition 2.2. This implies that

$$\lim_{p \searrow (d-1)/2} NCW(2p, I_d, I_d) = NCW(d-1, I_d, I_d)$$

exists by considering the Laplace transforms (this crucial remark is due to Mayerhofer (2010)). From Proposition 2.1 we have the result. \square

Proposition 2.3. Suppose $d \geq 3$. If $m(d-2, d-1, d)$ does not exist then $m(n, k, d)$ exists for no pairs (n, k) such that $0 \leq n < k < d$. If $m(d-2, d, d)$ does not exist then $m(n, d, d)$ exists for no n such that $0 \leq n \leq d-2$.

Proof. Suppose that $m(n, k, d)$ exists for some pair $0 \leq n < k < d$. We define $m'(dx)$ as the measure on $\overline{\mathcal{P}}_d$ with Laplace transform

$$\int_{\overline{\mathcal{P}}_d} e^{-\text{tr}(sx)} m'(dx) = \frac{1}{(\det s)^{\frac{d-n-2}{2}}} e^{\text{tr}[s^{-1}(I(d-1, d) - I(k, d))]}$$

Since the rank of $I(d-1, d) - I(k, d)$ is equal to $d-1-k$ and less than or equal to $d-n-2$ then m' exists by Propositions 2.1 and 2.2. Now we write the convolution

$$m(n, k, d) * m' = m(d-2, d-1, d)$$

which contradicts the non-existence of $m(d-2, d-1, d)$. Similarly, suppose that $m(d-2, d, d)$ does not exist and that there exists n such that $0 \leq n \leq d-2$ and such that $m(n, d, d)$ exists. Then $m(n, d, d) * m(d-2-n, 0, d) = m(d-2, d, d)$ also leads to a contradiction. \square

The idea of the proof of Proposition 2.3 is essentially due to Mayerhofer (2010). Here is now his important main result:

Proposition 2.4. If $d \geq 3$ the measure $m(d-2, d, d)$ does not exist.

Here is our main result:

Proposition 2.5. If $d \geq 3$ the measure $m(d-2, d-1, d)$ does not exist.

We will prove Proposition 2.5 in Section 6. In the remainder of the paper we develop the tools that lead us to this proof. They will also enable us to give a quick proof of Proposition 2.4. Let us emphasize the fact that Propositions 2.1 to 2.5 give a necessary and sufficient condition of existence of the distribution $NCW(2p, w, \Sigma)$. Suppose that $d \geq 3$. Given $\Sigma \in \mathcal{P}_d$, $w \in \overline{\mathcal{P}}_d$ and $p > 0$ then $NCW(2p, w, \Sigma)$ exists if and only if the following are satisfied

1. $p \in \Lambda_d$ defined by (3);
2. if $2p \geq d-1$ then $\text{rank } w$ is arbitrary;
3. if $2p = n \leq d-2$ then $\text{rank } w \leq n$.

Note that for $d=2$ the probability $NCW(2p, w, \Sigma)$ exists if and only if $2p \geq 1$.

3 Computation of $m(1, 2, 2)$

In this section we compute $m(1, 2, 2)$ using only calculus. We parameterize the cone $\overline{\mathcal{P}}_2$ by the cone of revolution $C = \{(x, y, z) \in \mathbb{R}^3; x \geq \sqrt{y^2 + z^2}\}$ using the map φ from C to $\overline{\mathcal{P}}_2$ defined by $(x, y, z) \mapsto \begin{bmatrix} x+y & z \\ z & x-y \end{bmatrix}$. Note that $\text{tr}[\varphi(a, b, c)\varphi(x, y, z)] = 2ax + 2by + 2cz$.

Proposition 3.1: If $C = \{(x, y, z) \in \mathbb{R}^3; x \geq \sqrt{y^2 + z^2}\}$ consider the positive measure μ on C such that for $(a, b, c) \in C$ we have

$$\frac{1}{\sqrt{a^2 - b^2 - c^2}} e^{\frac{2a}{a^2 - b^2 - c^2}} = \int_C e^{-2ax - 2by - 2cz} \mu(dx, dy, dz), \quad (8)$$

that is to say such that the image of μ by φ is $m(1, 2, 2)$. Then

$$\mu(dx, dy, dz) = s(dx, dy, dz) + f(x, y, z) \mathbf{1}_C(x, y, z) dx dy dz$$

where the singular part s is the image of the measure $g(2\sqrt{y^2 + z^2})dydz$ on \mathbb{R}^2 by the map $(y, z) \mapsto (x, y, z) = (\sqrt{y^2 + z^2}, y, z)$ with

$$g(z) = \frac{2}{\pi z} \cosh(2\sqrt{z})$$

and where for $(x, y, z) \in C$

$$f(x, y, z) = \frac{2}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(x^2 - y^2 - z^2)^k}{k!(k+1)!} \sum_{m=0}^{\infty} \frac{1}{\Gamma(m+2k+\frac{5}{2})} \frac{(2x)^m}{m!}.$$

Proof: Let $D = \frac{\partial}{\partial x}$. Recall the following differentiation formula.

Faà di Bruno formula . If $f(t)$ and $g(x)$ are functions with enough derivatives, then

$$D^n f(g(x)) = \sum \frac{n!}{k_1! \dots k_n!} (D^k f)(g(x)) \left(\frac{Dg(x)}{1!} \right)^{k_1} \dots \left(\frac{D^n g(x)}{n!} \right)^{k_n}, \quad (9)$$

where $k = k_1 + \dots + k_n$ and where the sum is taken on all integers $k_j \geq 0$ such that $k_1 + 2k_2 + \dots + nk_n = n$.

For a reference see for instance Roman (1980). We apply (9) to g defined by $x \mapsto x^2 - y^2 - z^2$ for fixed y, z and to $f(t) = t^n$. Noting that $D^3 g = 0$, we obtain

$$\frac{\partial^n}{\partial x^n} (x^2 - y^2 - z^2)^n = n!^2 \sum_{k_2=0}^{[n/2]} \frac{1}{k_2!} \times \frac{(x^2 - y^2 - z^2)^{k_2}}{k_2!} \times \frac{(2x)^{n-2k_2}}{(n-2k_2)!}. \quad (10)$$

We now recall that for $p > 1/2$ we have

$$\frac{1}{(a^2 - b^2 - c^2)^p} = \frac{2}{\sqrt{\pi}} \times \frac{1}{\Gamma(p)\Gamma(p-\frac{1}{2})} \int_C e^{-2ax-2by-2cz} (x^2 - y^2 - z^2)^{p-\frac{3}{2}} dx dy dz \quad (11)$$

The idea of the proof is to apply (11) for $p = n + \frac{3}{2}$ and to write

$$\begin{aligned} \frac{1}{\sqrt{a^2 - b^2 - c^2}} e^{\frac{2a}{a^2 - b^2 - c^2}} &= \frac{1}{\sqrt{a^2 - b^2 - c^2}} + \sum_{n=0}^{\infty} \frac{(2a)^{n+1}}{(n+1)!} \times \frac{1}{(a^2 - b^2 - c^2)^{n+\frac{3}{2}}} \\ &= \frac{1}{\sqrt{a^2 - b^2 - c^2}} \\ &+ \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(2a)^{n+1}}{(n+1)!n!\Gamma(n+\frac{3}{2})} \int_C e^{-2ax-2by-2cz} (x^2 - y^2 - z^2)^n dx dy dz \end{aligned} \quad (12)$$

Define

$$I_k(n) = (2a)^k \int_C e^{-2ax-2by-2cz} (x^2 - y^2 - z^2)^n dx dy dz$$

A first step is to observe that for $k = 0, 1, \dots, n$ we have

$$I_k(n) = \int_C e^{-2ax-2by-2cz} \frac{\partial^k}{\partial x^k} (x^2 - y^2 - z^2)^n dx dy dz. \quad (13)$$

Let us prove it by induction on k . It is true for $k = 0$. Suppose that it is true for $k < n$ and let us show that (13) is true for $k + 1$. Observe that for fixed (y, z) the root $\sqrt{y^2 + z^2}$ of the polynomial $x \mapsto (x^2 - y^2 - z^2)^n$ has order n and this implies that $\frac{\partial^k}{\partial x^k} (x^2 - y^2 - z^2)^n$ is zero for $x = \sqrt{y^2 + z^2}$.

Using this remark and integration by parts with $V(x) = e^{-2ax}$ and $U(x) = \frac{\partial^k}{\partial x^k}(x^2 - y^2 - z^2)^n$, we compute the following integral:

$$\int_{\sqrt{y^2+z^2}}^{\infty} 2ae^{-2ax} \frac{\partial^k}{\partial x^k}(x^2 - y^2 - z^2)^n dx = \int_{\sqrt{y^2+z^2}}^{\infty} e^{-2ax} \frac{\partial^{k+1}}{\partial x^{k+1}}(x^2 - y^2 - z^2)^n dx \quad (14)$$

With (14) we are in position to prove (13). We have

$$\begin{aligned} I_{k+1}(n) &= 2a \int_C e^{-2ax-2by-2cz} \frac{\partial^k}{\partial x^k}(x^2 - y^2 - z^2)^n dx dy dz \\ &= \int_{\mathbb{R}^2} e^{-2by-2cz} \left[\int_{\sqrt{y^2+z^2}}^{\infty} 2ae^{-2ax} \frac{\partial^k}{\partial x^k}(x^2 - y^2 - z^2)^n dx \right] dy dz \\ &= \int_{\mathbb{R}^2} e^{-2by-2cz} \left[\int_{\sqrt{y^2+z^2}}^{\infty} e^{-2ax} \frac{\partial^{k+1}}{\partial x^{k+1}}(x^2 - y^2 - z^2)^n dx \right] dy dz \\ &= \int_C e^{-2ax-2by-2cz} \frac{\partial^{k+1}}{\partial x^{k+1}}(x^2 - y^2 - z^2)^n dx dy dz \end{aligned}$$

which proves (13). We will need (13) only for $k = n$. The second step is to express $I_{n+1}(n)$ as the Laplace transform of a positive measure. We compute $I_n(n)$ using again an integration by parts. The new fact for $k = n$ is that the integrated part will not disappear and will provide a term for the singular measure s given in the statement of the theorem. This calculation of the integrated part will use (10). Taking $V(x) = -e^{-2ax}$ and $U(x) = \frac{\partial^n}{\partial x^n}(x^2 - y^2 - z^2)^n$, we have

$$\begin{aligned} I_{n+1}(n) = 2aI_n(n) &= \int_{\mathbb{R}^2} e^{-2by-2cz} \left[\int_{\sqrt{y^2+z^2}}^{\infty} 2ae^{-2ax} \frac{\partial^n}{\partial x^n}(x^2 - y^2 - z^2)^n dx \right] dy dz \\ &= \int_C e^{-2ax-2by-2cz} \frac{\partial^{n+1}}{\partial x^{n+1}}(x^2 - y^2 - z^2)^n dx dy dz \end{aligned} \quad (15)$$

$$+ n! \int_{\mathbb{R}^2} e^{-2by-2cz} [-e^{-2ax}(2x)^n]_{\sqrt{y^2+z^2}}^{\infty} dy dz \quad (16)$$

where (16) comes from (10) by keeping only the term $k_2 = 0$. We finally obtain

$$I_{n+1}(n) = \int_C e^{-2ax-2by-2cz} \frac{\partial^{n+1}}{\partial x^{n+1}}(x^2 - y^2 - z^2)^n dx dy dz + n! \int_{\mathbb{R}^2} e^{-2a\sqrt{y^2+z^2}-2by-2cz} (2\sqrt{y^2+z^2})^n dy dz.$$

Note that (12) is

$$\frac{1}{\sqrt{a^2 - b^2 - c^2}} e^{\frac{2a}{a^2 - b^2 - c^2}} = \frac{1}{\sqrt{a^2 - b^2 - c^2}} + \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{1}{(n+1)!n!\Gamma(n + \frac{3}{2})} I_{n+1}(n).$$

The last step of the proof is to represent the function on $C \setminus \partial C$ defined by $(a, b, c) \mapsto \frac{1}{\sqrt{a^2 - b^2 - c^2}}$ as a Laplace transform. Using the Gaussian integral in (17) we obtain

$$\frac{1}{\sqrt{a^2 - b^2 - c^2}} = \frac{2}{\pi} \int_{\mathbb{R}^2} e^{-2a(u^2+v^2)-2b(u^2-v^2)-4cuv} du dv \quad (17)$$

$$= \frac{2}{\pi} \int_{\mathbb{R}^2} e^{-2a\sqrt{y^2+z^2}-2by-2cz} (2\sqrt{y^2+z^2})^{-1} dy dz \quad (18)$$

To derive (18) observe that the map on $\{(u, v); u > 0\}$ defined by $y = u^2 - v^2$, $z = 2uv$ is a bijection with \mathbb{R}^2 ; the same is true with $\{(u, v); u < 0\}$. Furthermore $dy dz = 4(u^2 + v^2) du dv = 4\sqrt{y^2 + z^2} du dv$ and therefore $du dv = \frac{dy dz}{4\sqrt{y^2 + z^2}}$. All of this leads to (18).

Now we use $\Gamma(n + \frac{1}{2}) = \frac{(2n)!}{4^n n!} \sqrt{\pi}$ and we consider the function

$$g(t) = \frac{2}{\pi t} + \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{1}{(n+1)! \Gamma(n + \frac{3}{2})} t^n = \frac{2}{\pi t} \cosh(2\sqrt{t})$$

We then define the measure $s(dx, dy, dz)$ concentrated on the boundary ∂C of the cone to be the image of the measure

$$g(2\sqrt{y^2 + z^2}) dy dz = \frac{1}{\pi \sqrt{y^2 + z^2}} \cosh(2^{3/2}(y^2 + z^2)^{1/4}) dy dz \quad (19)$$

by the map $(y, z) \mapsto (x, y, z) = (\sqrt{y^2 + z^2}, y, z)$. This s will be the singular part of the measure.

We now concentrate on the absolutely continuous part. We will need the following formula, similar to (10) and also obtained by the Faà di Bruno formula (9):

$$\frac{\partial^n}{\partial x^n} (x^2 - y^2 - z^2)^{n-1} = n!(n-1)! \sum_{k_2=1}^{[n/2]} \frac{1}{(k_2-1)!} \times \frac{(x^2 - y^2 - z^2)^{k_2-1}}{k_2!} \times \frac{(2x)^{n-2k_2}}{(n-2k_2)!}. \quad (20)$$

The absolutely continuous part of $m(1, 2, 2)$ is given by the part of (12) defined on C and its density is

$$\begin{aligned} f(x, y, z) &= \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{1}{(n+1)! n! \Gamma(n + \frac{3}{2})} \frac{\partial^{n+1}}{\partial x^{n+1}} (x^2 - y^2 - z^2)^n \\ &= \frac{2}{\sqrt{\pi}} \sum_{n=2}^{\infty} \frac{1}{(n-1)! n! \Gamma(n + \frac{1}{2})} \frac{\partial^n}{\partial x^n} (x^2 - y^2 - z^2)^{n-1} \\ &= \frac{2}{\sqrt{\pi}} \sum_{n=2}^{\infty} \frac{1}{\Gamma(n + \frac{1}{2})} \sum_{k_2=1}^{[n/2]} \frac{1}{(k_2-1)!} \times \frac{(x^2 - y^2 - z^2)^{k_2-1}}{k_2!} \times \frac{(2x)^{n-2k_2}}{(n-2k_2)!} \\ &= \frac{2}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(x^2 - y^2 - z^2)^k}{k!(k+1)!} \sum_{m=0}^{\infty} \frac{1}{\Gamma(m + 2k + \frac{5}{2})} \frac{(2x)^m}{m!} \end{aligned}$$

□

Remark: The image by φ of the measure $s(dx, dy, dz)$ is concentrated on the set $S_1 \subset \overline{\mathcal{P}}_2$ of matrices of rank one. Any element of S_1 can be written as $u \begin{bmatrix} \lambda_1 & 0 \\ 0 & 0 \end{bmatrix} u^*$ where u is an orthogonal matrix of $\mathbb{O}(2)$ and $\lambda_1 > 0$. We can compute the image of $s(dx, dy, dz)$ by the map

$$\begin{bmatrix} x+y & z \\ z & x-y \end{bmatrix} = \begin{bmatrix} \sqrt{y^2 + z^2} + y & z \\ z & \sqrt{y^2 + z^2} - y \end{bmatrix} \mapsto \lambda_1 = 2\sqrt{y^2 + z^2}.$$

If $A_t = \{(x, y, z); 2\sqrt{y^2 + z^2} < t\}$, then using polar coordinates $y = \lambda_1 \cos \alpha, z = \lambda_1 \sin \alpha$ with Jacobian equal to $\frac{\lambda_1}{2}$, we have

$$s(A_t) = \int_{A_t} s(dx, dy, dz) = \int_{2\sqrt{y^2 + z^2} < t} g(2\sqrt{y^2 + z^2}) dy dz = \frac{\pi}{2} \int_0^t g(\lambda_1) \lambda_1 d\lambda_1.$$

Since $g(\lambda_1) = \frac{2}{\pi \lambda_1} \cosh 2\sqrt{\lambda_1}$, the image of s is

$$\cosh(2\sqrt{\lambda_1}) \mathbf{1}_{(0, \infty)}(\lambda_1) d\lambda_1. \quad (21)$$

Now an important observation is the following: consider the measure $m(1, 1, 1)(d\lambda)$ on $(0, \infty)$ whose Laplace transform is

$$\frac{1}{\sqrt{s}} e^{1/s} = \sum_{n=0}^{\infty} \frac{1}{n! s^{n+\frac{1}{2}}} = \int_0^{\infty} e^{-s\lambda} \sum_{n=0}^{\infty} \frac{\lambda^{n-\frac{1}{2}}}{n! \Gamma(n+\frac{1}{2})} d\lambda = \frac{1}{\sqrt{\pi}} \int_0^{\infty} e^{-s\lambda} \frac{1}{\sqrt{\lambda}} \cosh(2\sqrt{\lambda}) d\lambda.$$

As a consequence

$$m(1, 1, 1)(d\lambda) = \frac{1}{\sqrt{\pi}} \frac{1}{\sqrt{\lambda}} \cosh(2\sqrt{\lambda}) \mathbf{1}_{(0, \infty)}(\lambda) d\lambda \quad (22)$$

and one observes that $m(1, 1, 1)$ is quite close of (21). To summarize this remark, the singular part of $m(1, 2, 2)$ can be seen as the image of $\sqrt{\pi\lambda_1} m(1, 1, 1)(d\lambda_1) \otimes du$ by the map $(u, \lambda_1) \mapsto u \begin{bmatrix} \lambda_1 & 0 \\ 0 & 0 \end{bmatrix} u^*$ from $(0, \infty) \times \mathbb{O}(2)$ where du is the uniform probability on $\mathbb{O}(2)$. This is the key to the generalization of $m(1, 2, 2)$ to $m(d-1, d, d)$ for $d \geq 2$ done in Proposition 4.4 below.

4 Computation of the measure $m(d-1, d, d)$

Before stating Proposition 4.4 which describes $m(d-1, d, d)$ we have to fix some notations, recall a few facts about zonal functions and polynomials and prove three lemmas. The Lebesgue measure dx on the space of symmetric matrices of order d has the normalization associated to the Euclidean structure given by $\langle x, y \rangle = \text{tr}(xy)$. Let \mathcal{E}_d be the set of sequences $\kappa = (m_1, \dots, m_d)$ of d integers such that $m_1 \geq m_2 \geq \dots \geq m_d \geq 0$. If $\kappa \in \mathcal{E}_d$ we consider the two zonal polynomials $\Phi_{\kappa}(x) = \Phi_{m_1, \dots, m_d}(x)$ and $C_{\kappa}(x) = C_{\kappa}(I_d) \Phi_{\kappa}(x)$ where $C_{\kappa}(I_d)$ is defined below in (24). To introduce Φ_{κ} we consider

$$\Delta_{\kappa}(x) = \Delta_1(x)^{m_1-m_2} \Delta_2(x)^{m_2-m_3} \dots \Delta_{d-1}(x)^{m_{d-1}-m_d} \Delta_d(x)^{m_d}$$

where for $x = (x_{ij})_{1 \leq i, j \leq d}$ a real symmetric matrix, $\Delta_k(x) = \det(x_{ij})_{1 \leq i, j \leq k}$. The function Φ_{κ} is defined by

$$\Phi_{\kappa}(x) = \int_{\mathbb{O}(d)} \Delta_{\kappa}(uxu^*) du \quad (23)$$

where du is the Haar probability on the orthogonal group $\mathbb{O}(d)$. When $x \in \mathcal{P}_d$ definition (23) makes sense even when m_1, \dots, m_d are complex numbers. In that case $\Phi_{m_1, \dots, m_d}(x)$ is no longer a polynomial and is called a zonal function. To give the value of the constant $C_{\kappa}(I_d)$ we need the notations $\ell(\kappa) = \max\{j; m_j > 0\}$, $|\kappa| = m_1 + \dots + m_d$ and

$$\Gamma_d(z_1, \dots, z_d) = \prod_{j=1}^d \Gamma(z_j - \frac{j-1}{2})$$

defined for $z_j - \frac{j-1}{2} > 0$, $j = 1, \dots, d$. If p is a real number, we use the convention

$$\Gamma_d(z+p) = \Gamma_d(z_1+p, \dots, z_d+p).$$

If $\kappa \in \mathcal{E}_d$ and if $p > (d-1)/2$ we define the Pochhammer symbol $(p)_{\kappa} = \Gamma_d(\kappa+p)/\Gamma_d(p)$. If $\kappa \in \mathcal{E}_d$ the constant $C_{\kappa}(I_d)$ is given in Muirhead page 237 formula (38) by

$$C_{\kappa}(I_d) = C_{m_1, \dots, m_d}(I_d) = 2^{2|\kappa|} |\kappa|! \left(\frac{d}{2}\right)_{\kappa} \frac{\prod_{1 \leq i < j \leq \ell(\kappa)} (2m_i - 2m_j - i + j)}{\prod_{i=1}^{\ell(\kappa)} (2m_i + \ell(\kappa) - i)!}. \quad (24)$$

We never consider $C_{\kappa}(x)$ if $\kappa \notin \mathcal{E}_d$. The exact value of $C_{\kappa}(I_d)$ will be important in the proof of Proposition 4.4 when we shall need the formula (3) page 259 of Muirhead (1983):

$$e^{\text{tr } x} = \sum_{\kappa \in \mathcal{E}_d} \frac{1}{|\kappa|!} C_{\kappa}(x) \quad (25)$$

We are indebted to Jacques Faraut for the next lemma:

Lemma 4.1: If $x = \begin{bmatrix} x_1 & x_{12} \\ x_{21} & x_2 \end{bmatrix} \in \mathcal{P}_d$ we denote $[x]_1 = x_1 \in \mathcal{P}_{d-1}$. Then for all complex numbers m_1, \dots, m_d we have

$$\Phi_{m_1, \dots, m_d}(x) = (\det x)^{m_d} \int_{\mathbb{O}(d)} \Phi_{m_1, \dots, m_{d-1}}([uxu^*]_1) du.$$

Proof: Consider $v = \begin{bmatrix} v_1 & 0 \\ 0 & 1 \end{bmatrix} \in \mathbb{O}(d)$ where $v_1 \in \mathbb{O}(d-1)$. Observe that

$$[vyv^*]_1 = v_1[y]_1 v_1^* \quad (26)$$

We write

$$\Phi_{m_1, \dots, m_d}(x) = \int_{\mathbb{O}(d)} \Delta_{m_1, \dots, m_{d-1}, m_d}(uxu^*) du \quad (27)$$

$$= (\det x)^{m_d} \int_{\mathbb{O}(d)} \Delta_{m_1, \dots, m_{d-1}}([uxu^*]_1) du \quad (28)$$

$$= (\det x)^{m_d} \int_{\mathbb{O}(d)} \Delta_{m_1, \dots, m_{d-1}}([vuxu^*v^*]_1) du \quad (29)$$

$$= (\det x)^{m_d} \int_{\mathbb{O}(d)} \Delta_{m_1, \dots, m_{d-1}}(v_1[uxu^*v^*]_1 v_1^*) du \quad (30)$$

$$= (\det x)^{m_d} \int_{\mathbb{O}(d)} \left(\int_{\mathbb{O}(d-1)} \Delta_{m_1, \dots, m_{d-1}}(v_1[uxu^*v^*]_1 v_1^*) dv_1 \right) du \quad (31)$$

$$= (\det x)^{m_d} \int_{\mathbb{O}(d)} \Phi_{m_1, \dots, m_{d-1}}([uxu^*]_1) du \quad (32)$$

In this list (27) comes from the definition of $\Phi_\kappa(x)$, (28) separates the roles of $[uxu^*]_1$ and $\det(uxu^*) = \det x$ in the definition of $\Delta_\kappa(uxu^*)$, (29) uses the fact that du is the Haar measure, (30) applies (26) to $y = uxu^*$, (31) uses the fact that the Haar measure dv_1 of $\mathbb{O}(d-1)$ has mass 1, (32) comes from the definition of $\Phi_{m_1, \dots, m_{d-1}}(x)$. \square

Lemma 4.2: If $x \in \mathcal{P}_d$ then $\Phi_{m_1, \dots, m_d}(x^{-1}) = \Phi_{-m_d, \dots, -m_1}(x)$

Proof: Define $p \in \mathbb{O}(d)$ by

$$p = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 0 & \dots & 0 & 0 \end{bmatrix}$$

and define $\Delta_{m_1, \dots, m_{d-1}, m_d}^*(x) = \Delta_{m_1, \dots, m_{d-1}, m_d}(p x p^*)$. We can write

$$\int_{\mathbb{O}(d)} \Delta_{m_1, \dots, m_{d-1}, m_d}(ux^{-1}u^*) du = \int_{\mathbb{O}(d)} \Delta_{-m_d, \dots, -m_1}^*(u^*xu) du \quad (33)$$

$$= \int_{\mathbb{O}(d)} \Delta_{-m_d, \dots, -m_1}(uxu^*) du \quad (34)$$

In this list (33) comes from the FK formula Proposition VII.1.5 (ii) page 127 which says

$$\Delta_{m_1, \dots, m_{d-1}, m_d}(x^{-1}) = \Delta_{-m_d, \dots, -m_1}^*(x),$$

and (34) comes from the invariance of the Haar measure du on $\mathbb{O}(d)$ by $u \mapsto pu^*$. \square

Lemma 4.3: If $x \in \mathcal{P}_d$ then $\Phi_{m_1, \dots, m_d}(x)(\det x)^p = \Phi_{m_1+p, \dots, m_d+p}(x)$.

Proof: We have from the definition $\Phi_\kappa(x)(\det x)^p =$

$$\int_{\mathbb{O}(d)} \Delta_{m_1, \dots, m_d}(uxu^*)(\det uxu^*)^p du = \int_{\mathbb{O}(d)} \Delta_{m_1+p, \dots, m_d+p}(uxu^*) du = \Phi_{m_1+p, \dots, m_d+p}(x) \quad \square$$

Proposition 4.4. Define the singular measure $r(dt)$ on $\overline{\mathcal{P}_d}$ and concentrated on S_{d-1} as the image of the product measure

$$\frac{(\pi \det x)^{1/2}}{\Gamma(d/2)} m(d-1, d-1, d-1)(dx) \otimes du$$

by the map from $\mathcal{P}_{d-1} \times \mathbb{O}(d)$ to $\overline{\mathcal{P}_d}$ which is $(x, u) \mapsto t = u \begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix} u^* = u\tilde{x}u^*$. Define

$$f_d(t) = (\det t)^{-1} \left(\sum_{\kappa \in \mathcal{E}'_d} \frac{C_\kappa(t)}{|\kappa|! \Gamma_d(\kappa + \frac{d-1}{2})} \right).$$

where $\mathcal{E}'_d = \{\kappa \in \mathcal{E}_d ; m_d > 0\}$. Then

$$m(d-1, d, d)(dt) = r(dt) + f_d(t) \mathbf{1}_{\mathcal{P}_d}(t) dt.$$

Proof. The function $f_d(t)$ is a well defined analytic function around $x = 0$ since from the definition (23) of Φ_κ the polynomial $C_{m_1, \dots, m_d}(x)$ is divisible by $(\det x)^{m_d}$. Therefore $(\det x)^{-1} C_\kappa(x)$ is a polynomial when $\kappa \in \mathcal{E}'_d$. Recall the basic fact (see FK, Lemma XI.2.3 or Theorem XIV.3.1):

$$\int_{\mathcal{P}_d} e^{-\text{tr}(sx)} (\det x)^{p - \frac{d+1}{2}} \frac{\Phi_\kappa(x)}{\Gamma_d(\kappa + p)} dx = \Phi_\kappa(s^{-1}) (\det s)^{-p}. \quad (35)$$

Note that the choice of the suitable Lebesgue measure dx is crucial in (35). From (25) we know that, for $2p \geq d-1$, the Laplace transform of $m(2p, d, d)$ is

$$\int_{\mathcal{P}_d} e^{-\text{tr}(sx)} m(2p, d, d)(dx) = (\det s)^{-p} \sum_{\kappa \in \mathcal{E}_d} \frac{C_\kappa(s)}{|\kappa|!} \quad (36)$$

Observe that the Laplace transform of $f_d(t) \mathbf{1}_{\mathcal{P}_d}(t) dt$ is easily deduced from (35) and is

$$\int_{\mathcal{P}_d} e^{-\text{tr}(sx)} f_d(t) dt = (\det s)^{-\frac{d-1}{2}} \left(\sum_{\kappa \in \mathcal{E}'_d} \frac{C_\kappa(s^{-1})}{|\kappa|!} \right) \quad (37)$$

In (36) take $2p = d-1$. From the Laplace transform (37) our aim is therefore to prove that the Laplace transform of $r(dt)$ is

$$\int_{\mathcal{P}_d} e^{-\text{tr}(st)} r(dt) = (\det s)^{-\frac{d-1}{2}} \left(\sum_{\kappa \in \mathcal{E}_d \setminus \mathcal{E}'_d} \frac{C_\kappa(s^{-1})}{|\kappa|!} \right) = (\det s)^{-\frac{d-1}{2}} \left(\sum_{\kappa \in \mathcal{E}_{d-1}} \frac{C_{(\kappa, 0)}(s^{-1})}{|\kappa|!} \right) \quad (38)$$

To prove (38) we undertake the calculation of the Laplace transform of r . Observe first that (35) and (36) imply for $2p > d-1$

$$m(2p, d, d)(dx) = (\det x)^{p - \frac{d+1}{2}} \left(\sum_{\kappa \in \mathcal{E}_d} \frac{C_\kappa(x)}{|\kappa|! \Gamma_d(\kappa + p)} \right) \mathbf{1}_{\mathcal{P}_d}(x) dx. \quad (39)$$

In particular, in (39) let us replace d by $d - 1$ and do $2p = d - 1$. We get

$$(\det x)^{\frac{1}{2}} m(d-1, d-1, d-1)(dx) = \left(\sum_{\kappa \in \mathcal{E}_{d-1}} \frac{C_\kappa(x)}{|\kappa|! \Gamma_{d-1}(\kappa + \frac{d-1}{2})} \right) \mathbf{1}_{\mathcal{P}_{d-1}}(x) dx. \quad (40)$$

We can now write

$$\int_{\mathcal{P}_d} e^{-\text{tr}(st)} r(dt) = \frac{\pi^{1/2}}{\Gamma(d/2)} \int_{\mathbb{O}(d)} \left(\int_{\mathcal{P}_{d-1}} e^{-\text{tr}(su\tilde{x}u^*)} \sum_{\kappa \in \mathcal{E}_{d-1}} \frac{C_\kappa(x)}{|\kappa|! \Gamma_{d-1}(\kappa + \frac{d-1}{2})} dx \right) du \quad (41)$$

$$= \frac{\pi^{1/2}}{\Gamma(d/2)} \sum_{\kappa \in \mathcal{E}_{d-1}} \frac{C_\kappa(I_{d-1})}{|\kappa|!} \int_{\mathbb{O}(d)} \left(\int_{\mathcal{P}_{d-1}} e^{-\text{tr}([u^* su]_1 x)} \frac{\Phi_\kappa(x)}{\Gamma_{d-1}(\kappa + \frac{d-1}{2})} dx \right) du$$

$$= \frac{\pi^{1/2}}{\Gamma(d/2)} \sum_{\kappa \in \mathcal{E}_{d-1}} \frac{C_\kappa(I_{d-1})}{|\kappa|!} \int_{\mathbb{O}(d)} (\det[usu^*]_1^{-1})^{\frac{d}{2}} \Phi_\kappa([usu^*]_1^{-1}) du \quad (42)$$

$$= \frac{\pi^{1/2}}{\Gamma(d/2)} \sum_{\kappa \in \mathcal{E}_{d-1}} \frac{C_\kappa(I_{d-1})}{|\kappa|!} \int_{\mathbb{O}(d)} \Phi_{\kappa + \frac{d}{2}}([usu^*]_1^{-1}) du \quad (43)$$

$$= \frac{\pi^{1/2}}{\Gamma(d/2)} \sum_{\kappa \in \mathcal{E}_{d-1}} \frac{C_\kappa(I_{d-1})}{|\kappa|!} \int_{\mathbb{O}(d)} \Phi_{-m_{d-1}-\frac{d}{2}, \dots, -m_1-\frac{d}{2}}([usu^*]_1) du \quad (44)$$

$$= \frac{\pi^{1/2}}{\Gamma(d/2)} \sum_{\kappa \in \mathcal{E}_{d-1}} \frac{C_\kappa(I_{d-1})}{|\kappa|!} \Phi_{-m_{d-1}-\frac{d}{2}, \dots, -m_1-\frac{d}{2}, 0}(s) \quad (45)$$

$$= \frac{\pi^{1/2}}{\Gamma(d/2)} \sum_{\kappa \in \mathcal{E}_{d-1}} \frac{C_\kappa(I_{d-1})}{|\kappa|!} \Phi_{0, m_1+\frac{d}{2}, \dots, m_{d-1}+\frac{d}{2}}(s^{-1}) \quad (46)$$

$$= \frac{\pi^{1/2}}{\Gamma(d/2)} \sum_{\kappa \in \mathcal{E}_{d-1}} \frac{C_\kappa(I_{d-1})}{|\kappa|!} \Phi_{-\frac{d}{2}, m_1, \dots, m_{d-1}}(s^{-1}) (\det s^{-1})^{\frac{d}{2}} \quad (47)$$

$$= \frac{\pi^{1/2}}{\Gamma(d/2)} \sum_{\kappa \in \mathcal{E}_{d-1}} \frac{C_\kappa(I_{d-1})}{|\kappa|!} \Phi_{m_1, \dots, m_{d-1}, 0}(s^{-1}) (\det s^{-1})^{\frac{d-1}{2}} \quad (48)$$

$$= (\det s)^{-\frac{d-1}{2}} \sum_{\kappa \in \mathcal{E}_{d-1}} \frac{C_{(\kappa, 0)}(s^{-1})}{|\kappa|!} \quad (49)$$

In this list equality (41) comes from (40), equality (42) comes from (35) and equalities (43) and (47) come from Lemma 4.3. In identities following (44) we have replaced κ by (m_1, \dots, m_{d-1}) for clarity. Formulas (44) and (46) come from Lemma 4.2, and (45) comes from Lemma 4.1. The proof of (48) is more involved and is a consequence of formula (iii) of Theorem XIV 3.1 of FK where we replace (d, r, λ, μ) respectively by $1, d$ and

$$\begin{aligned} \lambda &= (m_1 + \frac{d-1}{4}, m_2 + \frac{d-3}{4}, \dots, m_{d-1} - \frac{d-3}{4}, -\frac{d-1}{4}), \\ \mu &= (-\frac{d-1}{4}, m_1 + \frac{d-1}{4}, m_2 + \frac{d-3}{4}, \dots, m_{d-1} - \frac{d-3}{4}) \end{aligned}$$

The fact that μ is a permutation of λ and the reference above imply (48). The proof of (49) comes from $\frac{\pi^{1/2}}{\Gamma(d/2)} C_\kappa(I_{d-1}) = C_{\kappa, 0}(I_d)$ implied by formula (24). An important point for this is simply $\ell(\kappa) = \ell(\kappa, 0)$. Finally (49) proves (38) and Proposition 4.4 itself. \square

5 Convolution lemmas in the cone $\overline{\mathcal{P}_d}$

We give the proof of Lemma 5.1 below though it is certainly a known fact.

Lemma 5.1: In a Euclidean space E of dimension d let us fix a linear subspace F of dimension n . We choose randomly a linear subspace G of dimension $k \leq d - n$ with the uniform distribution, that is the unique distribution on G such that $G \sim uG$ for all $u \in \mathbb{O}(d)$. Then $\Pr(G \cap F \neq \{0\}) = 0$.

Proof: It is enough to prove the lemma for $E = \mathbb{R}^d$, $F = \{0\} \times \mathbb{R}^{d-n}$ and $k = d - n$. Let Z_1, \dots, Z_n be i.i.d. random variables in \mathbb{R}^d following the standard Gaussian distribution $N(0, I_d)$. Let G be the random linear subspace of E generated by the vectors Z_1, \dots, Z_n . Since for all $u \in \mathbb{O}(d)$ we have $(uZ_1, \dots, uZ_n) \sim (Z_1, \dots, Z_n)$ clearly $G \sim uG$ and G has the uniform distribution. Introduce the matrix

$$M = [Z_1, \dots, Z_n] = (Z_{ij})_{1 \leq i \leq d, 1 \leq j \leq n}$$

whose columns are the vectors Z_1, \dots, Z_n . Then $x_1 Z_1 + \dots + x_n Z_n = MX$ where $X = (x_1, \dots, x_n)^*$. Now $G \cap F \neq \{0\}$ if and only if there exists a non-zero X such that the n first elements of MX are zero. In other terms, considering the square matrix M_1 of order n defined by $M_1 = (Z_{ij})_{1 \leq i, j \leq n}$, we have that $G \cap F \neq \{0\}$ if and only if there exists a non-zero X such that $M_1 X = 0$, which happens if and only if $\det M_1 = 0$. Since the n^2 entries of the matrix M_1 are independent $N(0, 1)$ variables, the event $\det M_1 = 0$ has probability zero and this proves the lemma. \square

In the sequel we denote by S_b the set of $x \in \overline{\mathcal{P}_d}$ with $b = \text{rank } x = 0, \dots, d$.

Lemma 5.2: Let Y be a random variable in S_b and assume that $uYu^* \sim Y$ for all u in the orthogonal group $\mathbb{O}(d)$. Let $x_0 \in S_a$. Then $x_0 + Y$ is concentrated on S_{a+b} if $a + b \leq d$ or on $S_d = \mathcal{P}_d$ if $a + b \geq d$. Furthermore if $x_0 \in S_c$, and if $x_0 + Y$ is concentrated on S_{a+b} with $a + b < d$ then $c = a$.

Remark: If $a + b = d$ and if $x_0 + Y$ is concentrated on $S_{a+b} = S_d$, x_0 could be on any S_c with $a \leq c \leq k$.

Proof: Apply Lemma 5.1 to $F = x_0 \mathbb{R}^d$ and to $G = Y \mathbb{R}^d$. Then $\dim(F + G) = a + b$ if $a + b \leq d$. This implies that $\text{rank}(x_0 + Y) \leq a + b$. To see that $\text{rank}(x_0 + Y) = a + b$ suppose that $(x_0 + Y) \mathbb{R}^d \neq E = F + G$. Let x'_0 and Y' be the restriction of the endomorphisms x_0 and Y to the linear space E . Since x_0 and Y are symmetric, this implies that $x'_0 E = F$ and $Y' E = G$. Since $(x'_0 + Y') E \neq E$ there exists $v \in E \setminus \{0\}$ which is orthogonal to $(x'_0 + Y') E$ and this implies that $(x'_0 + Y') v = 0$. Since $x'_0 v \in F$ and $Y' v \in G$ and since $F \cap G = \{0\}$ this implies that $x'_0 v = Y' v = 0$, and v is in $\text{Ker}(x'_0) \cap \text{Ker}(Y')$. But since $F \oplus G = E$ (a direct sum, not necessarily an orthogonal one) we have also $\text{Ker}(x'_0) \oplus \text{Ker}(Y') = E$ which implies $\text{Ker}(x'_0) \cap \text{Ker}(Y') = \{0\}$. Thus $v = 0$, a contradiction. Finally $(x_0 + Y) \mathbb{R}^d = E = F + G$ and $\text{rank}(x_0 + Y) = a + b$.

If $a + b > d$ then F contains a subspace of dimension $d - b$ and $\dim(F + G) = d$. Suppose now that $x_0 \in S_c$, and that $x_0 + Y$ is concentrated on S_{a+b} with $a + b < k$. If $c + b < d$ then, by the first part of the lemma, $x_0 + Y$ is concentrated on S_{c+b} . But $S_{a+b} = S_{c+b}$ implies $c = a$. If $c + b \geq k$ then, by the first part of the lemma again, $x_0 + Y$ is concentrated on S_d . This is impossible since $S_{a+b} \neq S_d$. \square

Lemma 5.3: Let μ and ν be positive measures on $\overline{\mathcal{P}_d}$ such that ν is concentrated on S_b and ν is invariant by the transformations $x \mapsto uxu^*$, $u \in \mathbb{O}(d)$. Let $a = 0, \dots, d - 2$ such that $a + b < d$. Then $\mu * \nu$ is concentrated on S_{a+b} if and only if μ is concentrated on S_a . Furthermore $\mu * \nu$ is concentrated on $S_d = \mathcal{P}_d$ if μ is concentrated on S_{d-b} .

Proof: \Rightarrow For $y_0 \in S_b$ consider the distribution $K_{y_0}(dy)$ on S_b of the random variable Uy_0U^* where U is uniformly distributed on the orthogonal group $\mathbb{O}(k)$. Let D_b the set of diagonal elements y_0 of

S_b of the form $y_0 = \text{diag}(\lambda_1, \dots, \lambda_b, 0, \dots, 0)$ such that $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_b > 0$. Then there exists a unique positive measure ν_0 on D_b such that the following desintegration holds

$$\nu(dy) = \int_{D_b} \nu_0(dy_0) K_{y_0}(dy).$$

It follows that

$$\begin{aligned} (\mu * \nu)(dx) &= \int_{S_b} \nu(dy) \mu(dx - y) = \int_{S_b} \mu(dx - y) \int_{D_b} \nu_0(dy_0) K_{y_0}(dy) \\ &= \int_{D_b} \nu_0(dy_0) \int_{S_b} \mu(dx - y) K_{y_0}(dy) \end{aligned}$$

Therefore the measure $\mu * K_{y_0}$ is concentrated on S_{a+b} for ν_0 almost all $y_0 \in D_b$. From Lemma 5.2 this implies that μ is concentrated on S_a .

\Leftarrow If μ is concentrated on S_a with $a + b \leq k$ it is an easy consequence of Lemma 5.2 that $\mu * \nu$ is concentrated on S_{a+b} . \square

Lemma 5.4: Let $a, b \in \{1, \dots, d-1\}$ such that $a + b < d$. If $m(a, a+b, d)$ exists, it is concentrated on S_a .

Proof: From the Laplace transforms we know that $m(a, a+b, d) * m(b, 0, d) = m(a+b, a+b, d)$. From Proposition 2.2 we know that $m(a+b, a+b, d)$ is concentrated on S_{a+b} . Since the Laplace transform of $m(b, 0, d)$ is $(\det s)^{-b/2}$ we know that $m(b, 0, d)$ is invariant by the transformations $x \mapsto u x u^*$ for any $u \in \mathbb{O}(d)$. By Lemma 5.3 we deduce that $m(a, a+b, d)$ is concentrated on S_a if it exists. \square

6 $m(d-2, d-1, d)$ and $m(d-2, d, d)$ do not exist for $d \geq 3$

In this section we prove Propositions 2.4 and 2.5.

Proof of Proposition 2.4. Suppose that $m(d-2, d, d)$ exists. By Lemma 5.4 the measure $m(d-2, d, d)$ is concentrated on S_{d-2} . By Lemma 5.3 the convolution

$$m(d-2, d, d) * m(1, 0, d) = m(d-1, d, d)$$

is concentrated on S_{d-1} . This contradicts Proposition 4.4. \square

Proof of Proposition 2.5. Suppose that $m(d-2, d-1, d)$ exists. By Lemma 5.4 the measure $m(d-2, d-1, d)$ is concentrated on S_{d-2} . Therefore there exists a positive measure $m(dy) = m(dy_1, \dots, dy_{d-2})$ on $\mathbb{R}^{d(d-2)} = \mathbb{R}^d \times \dots \times \mathbb{R}^d$ such that for all $s \in \mathcal{P}_d$ we have

$$\int_{\mathbb{R}^{d(d-2)}} e^{-(y_1^* s y_1 + \dots + y_{d-2}^* s y_{d-2})} m(dy) = \frac{1}{\det s^{(d-2)/2}} e^{\text{tr}(s^{-1} I_{(d-1, d)})}. \quad (50)$$

We write more conveniently the elements $y = (y_1, \dots, y_{d-2})$ with the help of the transposed matrix $y^* = (y_{i,j})$ with $d-2$ rows y_1^*, \dots, y_{d-2}^* and d columns c_1, \dots, c_d

$$y^* = \begin{bmatrix} y_1^* \\ \dots \\ y_{d-2}^* \end{bmatrix} = \begin{bmatrix} y_{1,1} & \dots & y_{1,d} \\ \dots & \dots & \dots \\ y_{d-2,1} & \dots & y_{d-2,d} \end{bmatrix} = [c_1, \dots, c_d].$$

With this notation introduce the Gram matrix

$$G(c) = G(c_1, \dots, c_d) = (\langle c_j, c_k \rangle)_{1 \leq j, k \leq d}$$

and denote by $m(dc)$ what we denoted by $m(dy)$ before. We get

$$\int_{\mathbb{R}^{d(d-2)}} e^{-\text{tr}(sG(c))} m(dc) = \frac{1}{\det s^{(d-2)/2}} e^{\text{tr}(s^{-1}I^{(d-1,d)})}. \quad (51)$$

Equality (51) means that $m(d-2, d-1, d)(dx)$ is the image of $m(dc)$ by $c \mapsto x = G(c)$.

Now in (51) we choose $s = \text{diag}(1, s_1)$ where s_1 is a symmetric positive definite matrix of order $d-1$. We also desintegrate $m(dc)$ by introducing a probability kernel $K(c_2, \dots, c_d; dc_1)$ and a positive measure $m_1(dc_2, \dots, dc_d)$ such that

$$e^{-\|c_1\|^2} m(dc_1, dc_2, \dots, dc_d) = m_1(dc_2, \dots, dc_d) K(c_2, \dots, c_d; dc_1)$$

With these notations we can write

$$\begin{aligned} & \frac{1}{\det s_1^{(d-2)/2}} e^{\text{tr}(s_1^{-1})} = \int_{\mathbb{R}^{d(d-2)}} e^{-\text{tr}(sG(c))} m(dc) \\ &= \int_{\mathbb{R}^{d(d-2)}} e^{-\|c_1\|^2} e^{-\text{tr}(s_1 G(c_2, \dots, c_d))} m(dc_1, dc_2, \dots, dc_d) \\ &= \int_{\mathbb{R}^{(d-1)(d-2)}} e^{-\text{tr}(s_1 G(c_2, \dots, c_d))} \left(\int_{\mathbb{R}^{d-2}} K(c_2, \dots, c_d; dc_1) \right) m_1(dc_2, \dots, dc_d) \\ &= \int_{\mathbb{R}^{(d-1)(d-2)}} e^{-\text{tr}(s_1 G(c_2, \dots, c_d))} m_1(dc_2, \dots, dc_d) \end{aligned}$$

since K is a probability kernel. The last equality says that the image of $m_1(dc_2, \dots, dc_d)$ by the map $(c_2, \dots, c_d) \mapsto x = G(c_2, \dots, c_d)$ is nothing but $m(d-2, d-1, d-1)(dx)$. Denote $G_2 = G(c_2, \dots, c_d)$ for simplicity. Since c_2, \dots, c_d are vectors of a Euclidean space of dimension $d-2$ the rank of G_2 is less than or equal to $d-2$. To prove this elementary fact of linear algebra we use $G_2 \in \overline{\mathcal{P}_{d-1}}$. This implies that if $x = (x_2, \dots, x_d)^*$ then $G_2 x = 0$ if and only if $x^* G_2 x = 0$. Since $x^* G_2 x = \|\sum_{i=2}^d x_i c_i\|^2$ the linear space of $x \in \mathbb{R}^{d-1}$ such that $\sum_{i=2}^d x_i c_i = 0$ has at least dimension 1, the kernel of the endomorphism of \mathbb{R}^{d-1} with matrix G_2 has at least dimension 1 and its image has at most dimension $d-2$. This contradicts Proposition 4.4 which says that $m(d-2, d-1, d-1)$ has an absolutely continuous part and therefore charges matrices with rank $d-1$. \square

7 References

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